BENDING OF SHELLS WITH POSITIVE CURVATURE UNDER A CONCENTRATED LOAD

PMM Vol. 31, No. 5, 1967, pp. 883-886
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(Received January 8, 1967)

The study is concerned with single-layer, convex, isotropic and anisotropic shells which permit a breakdown of the state of stress into a momentless one and boundary effects [1 and 2]; an approximate value is obtained from the normal deflection w at the point of application of a concentrated load acting in the direction normal to the middle surface. The solution is obtained by asymptotic integration of the equations of shell theory, whose application to the effects of concentrated loads on isotropic shells was developed in [3].

1. Anisotropic shell (general anisotropy). Assume that the solution of the membrane theory for a shell with a concentrated load is known (e.g. for shells described by second order surfaces the result is easily obtained by the method developed in [1]. At the point of application of the load, these solutions have singularities of a higher order than a general shell theory with moments would have [4]. The incompatibility thus obtained, as shown in [3], may be eliminated with the aid of rapidly varying solutions of the boundary effect type, which are of a local character, i.e. their essential contribution is confined to a sufficiently small neighborhood of the applied load. Such solutions for isotropic shells were obtained in [3], and were called local. These will be generalized here to the anisotropic shell. It is known [1] that the equation for rapidly varying solutions coincides with the equation for a sloping shell. However, it should be recalled that in obtaining the rapidly varying solutions it is necessary to neglect the slowly varying solutions. Let us write this equation [2]

$$L_1(D_{ik})\, w + \Delta_r \varphi = Z, \qquad L_2(A_{ik})\, \varphi - \Delta_r w = 0 \qquad \left(\Delta_r = \frac{1}{R_2 A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{R_1 B^2}\frac{\partial^2}{\partial\beta^2}\right) \quad (1)$$

$$L_1(D_{ik}) = \frac{D_{11}}{A^4}\frac{\partial^4}{\partial\alpha^4} + 4\frac{D_{16}}{A^3 B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} + 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} +$$
$$+ 4\frac{D_{26}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3} + \frac{D_{22}}{B^4}\frac{\partial^4}{\partial\beta^4}$$

$$L_2(A_{ik}) = \frac{A_{22}}{A^4}\frac{\partial^4}{\partial\alpha^4} - 2\frac{A_{26}}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} + \frac{A_{66} + 2A_{12}}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} - 2\frac{A_{16}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3} + \frac{A_{11}}{B^4}\frac{\partial^4}{\partial\beta^4}$$

$$A_{11} = (C_{22}C_{66} - C_{26}^2)\,\Omega^{-1}, \qquad A_{22} = (C_{11}C_{66} - C_{16}^2)\,\Omega^{-1}, \qquad C_{ik} = hB_{ik}$$
$$A_{12} = (C_{16}C_{26} - C_{12}C_{66})\,\Omega^{-1}, \qquad A_{66} = (C_{11}C_{22} - C_{12}^2)\,\Omega^{-1}$$
$$A_{16} = (C_{12}C_{26} - C_{16}C_{22})\,\Omega^{-1}, \qquad A_{26} = (C_{12}C_{16} - C_{26}C_{11})\,\Omega^{-1}, \qquad D_{ik} = {}^1/_{12}h^3B_{ik}$$
$$\Omega = (C_{11}C_{22} - C_{12}^2)\,C_{66} + 2C_{12}C_{16}C_{26} - C_{11}C_{26}^2 - C_{22}C_{16}^2$$

Here, φ is a stress function; B_{ik} are elastic constants [2]; h is the shell thickness: A B, R_1 and R_2 are the coefficients of the first quadratic form and the radii of curvature of the surface. Note that A, B, R_1, R_2 and B_{ik} are either constants or slowly varying functions which may be taken as constants with their values given at the point of application of the load. Then (1) may be reduced to the single Eq.

$$L_1 (D_{ik}) L_2 (A_{ik}) w + \Delta^2_r w = L_2 (A_{ik}) Z \tag{2}$$

For a normal, concentrated load P, the corresponding expression for Z is given by [4]

$$Z = P (AB)^{-1} \delta (\alpha_0, \beta_0)$$

Here, δ is the Dirac delta function.

We will construct a local solution to this equation, following the method in [3]. Introduce the nondimensional coordinates x and y

$$x = A (\alpha - \alpha_0) R_1^{-1/2} R_2^{-1/2}, \ y = B (\beta - \beta_0) \ R_1^{-1/2} \ R_2^{-1/2}$$

Then, as shown in [5], Z takes the form

$$Z = PR_1^{-1} R_2^{-1} \delta$$

The transformed Eq. (2) now becomes

$$R_1^{-1} R_2^{-1} \ L_1' L_2' w + \Delta_1^2 w = PL_2'\delta \tag{3}$$

Here, L_1', L_2' and Δ_1 are the transformed operators

$$L_1' = R_1^2 R_2^2 L_1 (D_{ik}), \qquad L_2' = R_1^2 R_2^2 L (A_{ik}), \qquad \Delta_1 = a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2}, \qquad a^2 = b^{-2} = \left(\frac{R_1}{R^2}\right)^{1/2}$$

To obtain a local solution, it is convenient to shift to a plane wave form in terms of ξ, since the solution to (3) may then be sought in the form of functions of ξ

$$\xi = x \cos\varphi + y \sin \varphi = r \cos (\varphi - \varphi_0), \qquad x = r \cos \varphi_0, \qquad y = r \sin \varphi_0$$

where φ is a parameter whose range is $[0, 2\pi)$. Indeed, the δ function has a plane wave representation [5]

$$\delta = -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{d\varphi}{\xi^2}$$

Substituting the above expression into (3), we obtain

$$\frac{1}{R_1 R_2} L_1' L_2' w + \Delta_1^2 w = -\frac{P}{4\pi^2} L_2 \int_0^{2\pi} \frac{\partial\varphi}{\xi^2}$$

We seek a solution to this equation in the form

$$w = \int_0^{2\pi} \Phi (\xi) \, d\varphi \tag{4}$$

Then Φ must satisfy the ordinary differential equation with constant coefficients

$$h_1^2 \frac{d^8\Phi}{d\xi^8} + t^2 \frac{d^4\Phi}{d\xi^4} = -\frac{Pl_2}{4\pi^2} \frac{d^4}{d\xi^4} \frac{1}{\xi^2} \qquad \left(h_1 = \frac{l_1 l_2}{R_1 R_2}, \ t = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi\right) \tag{5}$$

$$l_1 = D_{11} \cos^4\varphi + 4D_{16} \cos^3 \varphi \sin\varphi + 2 (D_{12} + 2D_{66}) \sin^2 \varphi \cos^2 \varphi + \\ + 4D_{26} \cos \varphi \sin^3 \varphi + D_{22} \sin^4 \varphi$$

$$l_2 = A_{22} \cos^4 \varphi - 2A_{26} \cos^3 \varphi \sin \varphi + (A_{26} + 4A_{12}) \cos^2 \varphi \sin^2 \varphi - \\ - 2A_{16} \cos \varphi \sin^3 \varphi + A_{11} \sin^4 \varphi$$

The solution to (5) may be represented as follows:

$$\Phi = + Pl_2 \frac{d^4\Phi^1}{d\xi^4} \tag{6}$$

where Φ^1 is a solution to

$$h_1^2 \frac{d^8\Phi^1}{d\xi^8} + t^2 \frac{d^4\Phi^1}{d\xi^4} = -\frac{1}{4\pi^2\xi^2} \tag{7}$$

An equation of similar form was investigated in [3], and it was shown how one may obtain a local solution from the general solution. The difference lies only in the fact that here, h_1 depends on φ whereas in [3] h_1 was constant. The solution to [7] is equal to the sum of solutions to the following Eqs. (this may be verified by substitution):

$$t^2 \frac{d^4\Phi_0^1}{d\xi^4} = -\frac{1}{4\pi^2\xi^2}, \qquad t^2 \frac{d^4\Phi_1^1}{d\xi^4} + \frac{t^4}{h_1^2}\Phi_1^1 = \frac{1}{4\pi\xi^2} \tag{8}$$

Eq. (7) has been decomposed into two: the first defines a slowly varying solution; the second, a rapidly varying solution. The second solution, which decreases for $\xi \to \pm\infty$, has the form

$$\Phi_1^1 = \frac{h_1}{16\pi^2 t^3} \operatorname{Im}\left[e^{-z}\operatorname{Ei}(z) + e^{z}\operatorname{Ei}(-z)\right], \qquad z = (1+i)\left(\frac{t}{2h_1}\right)^{1/2}\xi$$

The local w displacement, which will be denoted by w_0^3 as in [3] and which corresponds to Φ_1^1 has the form (according to (6) and (4))

$$w_0^3 = P\int_0^{2\pi} l_2 \frac{d^4\Phi_1^1}{d\xi^4}\, d\varphi \tag{9}$$

The solution (9) decreases with an increase in distance from the point ($x = 0$, $y = 0$).

The complete shell deflection consists of the combination of the membrane solution plus w_0^3. We will obtain its value at the point of application of the load.

The integrand in (9) may be transformed with the aid of (8) into

$$w_0^3 = -P\int_0^{2\pi}\left[\frac{t^2 l_2}{h_1^2}\Phi_1^1 - \frac{l_2}{4t^2\pi^2\xi^2}\right] d\varphi \tag{10}$$

It is known that the deflection under a concentrated load acting in a direction normal to the middle surface is finite. For isotropic shells, this was shown, for example, in [4]. The assertion holds for anisotropic shells as well. The deflection w^0 obtained from the membrane theory is infinite, with a singularity of order $(1/r^2)$. This may be shown with the aid of (3). Setting the shell thickness $h = 0$ in (3) corresponds to setting the operator $L_1 = 0$, i.e. the resultant equation corresponds to the membrane theory

$$\Delta_1^2 w^0 = PL_2\delta \tag{11}$$

The principal singularity of w^0 in the neighborhood of the singular point may be obtained by the previously discussed method of plane waves. We cite the final result

$$w^0 = -\frac{P}{4\pi^2}\int_0^{2\pi}\frac{l_2 d\varphi}{t^2\xi^2} = -\frac{P}{4\pi^2 r^2}\int_0^{2\pi}\frac{l_2\, d\varphi}{t^2\cos^2(\varphi-\varphi_0)} \tag{12}$$

Such is the principal singularity of the membrane theory. It can be seen that upon combining with the local solution (10), this singularity is eliminated

$$w = w^0 + w_0^3 = -P\int_0^{2\pi}\frac{l_2 t^2}{h_1^2}\Phi_1^1\, d\varphi \tag{13}$$

Since the general deflection is finite, all singularities of the membrane solution which yield an infinite deflection must be eliminated by combining with the local solution. Here we show that the first approximation of the local solution eliminates the principal singularity of the membrane solution; subsequent approximations eliminate the other singularities (of the type $1/r$, $\ln r$, etc.).

The finite part of the total deflection consists of the sum of the finite parts of the membrane and local solutions. In order to obtain the finite part of (13), it is necessary to expand it into a series in the neighborhood of the singular point, and the zeroth term of the series yields a first approximation of the finite part of the local solution.

The series for the integrand in (13) for the case of ($\ell_2/h_1^2 = 1$) is given in [3]. Utilizing this series expansion and adapting it to the case of $\ell_2/h_1^2 \neq 1$, we have

$$w = \ln r \sum_0^\infty P_{4k+2}(x, y) + \sum_0^\infty Q_{4k+2}(x, y) + \sum_0^\infty R_{4k}(x, y)$$

Here, P_i, Q_i and R_i are homogeneous polinomials of degree i. Clearly, the only nonzero contribution is made by the first term of the third series. This essential term may be written as

$$w_0^3(0, 0) = R_0 = P\frac{\sqrt{R_1R_2}}{32\pi}\int_0^{2\pi}\left(\frac{l_2}{l_1}\right)^{1/2}\frac{d\varphi}{t} \tag{14}$$

Here, the expressions for ℓ_1, ℓ_2 and t are as given in (5).

An estimate of the order of magnitude of the deflection in terms of the shell thickness $h_0 = hR_1^{-1/2}R_2^{-1/2}$, obtained with the aid of (1), yields

$$w_0^3(0, 0) \approx h_0^{-2}$$

An order of magnitude estimate of the finite part of the deflection given by the membrane theory may be obtained with the aid of (1) and (11). The result is $w^\circ(0, 0) \sim h_0^{-1}$. This quantity is negligible in comparison with w_0^3. Postulating that subsequent terms in the approximation for the total deflection, as obtained by asymptotic integration, are smaller by a factor of $\sqrt{h_0}$, as is the case in the boundary effect theory [1], then (14) may be considered as the total deflection of an anisotropic shell subjected to a concentrated load. An exact evaluation of the integral in (14) for the general case is cumbersome or even impossible.

It should be noted that (14) is valid only if the applied load is sufficiently far from the shell boundary so that the boundary effect does not influence the deflection at the point of application of the load, and, in turn, the local solution is sufficiently small at the boundary so that the conditions at the boundary are not sufficiently effected to cause an additional change at the point of application of the load. Such a distance may be taken as two to three times the boundary layer.

From the above discussion it is clear that, as a first approximation, the magnitude of the shell deflection is independent of the boundary conditions or boundary shape, and is determined by the radii of curvature at the point of application of the load, the material constants of the shell and shell thickness, i.e. the shell material and local geometry.

The deflection of a shell of variable thickness, if the thickness varies slowly, may be obtained from (14) by replacing h by $h(\alpha_0, \beta_0)$.

2. Isotropic shell. The magnitude of the deflection in this case is obtained as a particular case of the above. If the material is isotropic, (14) reduces to the following:

$$w(0, 0) = 4P\int_0^{2\pi}\frac{\sqrt{3(1-\sigma^2)R_1R_2}}{Eh^2t}d\varphi$$

The integral may be evaluated, and the deflection becomes

$$w(0, 0) = \frac{P}{4Eh^2} \sqrt{3(1-\sigma^2) R_1 R_2}$$

As first approximation, this expression for the deflection coincides with the exact expression obtained in [6].

The author is grateful to A. L. Gol'denveizer for his comments on the work.

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Translated by H. H.